

UCLA/14/TEP/110

# The Ultraviolet Critical Dimension of Half-Maximal Supergravity at Three Loops

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## Abstract

We determine the minimum dimension at which a divergence first occurs in half-maximal 16-supercharge supergravity at three loops. For the four-point amplitudes, we find that the critical dimension is  $D = 14/3$  and give the explicit form of the divergence. We also give the divergence of an additional piece that gives us access to theories with higher degrees of supersymmetry, in particular 20- and 32-supercharge supergravities. We give the divergences of half-maximal supergravity when matter vector multiplets are included as well. Explicit forms of various divergences at one and two loops are also given.

PACS numbers: 04.65.+e, 11.15.Bt, 11.25.Db, 12.60.Jv

## I. INTRODUCTION

The study of ultraviolet divergences in gravity theories has a long history dating back to the work of 't Hooft and Veltman [1]. Today, consensus opinion holds that all supergravity theories are necessarily ultraviolet divergent at some finite loop order. Indeed, recent studies of supergravity symmetries find that four-dimensional  $\mathcal{N} = 4$ ,  $\mathcal{N} = 5$  and  $\mathcal{N} = 8$  supergravity theories [2, 3] appear to have valid counterterms at three, four and seven loops, respectively, suggesting that these theories will likely diverge at these loop orders [4, 5]. However, more recent calculations demonstrate the existence of a new type of nontrivial multiloop ultraviolet cancellation called “enhanced cancellation” [6]. These cancellations are not accounted for in analyses based on standard-symmetry arguments. Enhanced cancellations have been explicitly exhibited in calculations in four-dimensional pure  $\mathcal{N} = 4$  and  $\mathcal{N} = 5$  supergravities where the expected divergences at three and four loops, respectively, are in fact not present [6, 7]. These calculations are made possible by the duality between color and kinematics [8, 9] and advanced integral reductions, as implemented in FIRE [10]. (See also Ref. [11] for a string-theory understanding of the finiteness in  $\mathcal{N} = 4$  supergravity.)

Enhanced cancellations are a supergravity phenomenon distinct from cancellations that control the leading ultraviolet behavior of (supersymmetric) gauge theory. Unlike gauge theory, an analysis of unitarity cuts in supergravity shows that enhanced cancellations cannot be made manifest diagram by diagram in any covariant local diagrammatic formalism based on ordinary Feynman propagators [6]. For  $\mathcal{N} = 8$  supergravity, the diagram-by-diagram power counting of maximal unitarity cuts is identical to the earlier power-counting analysis of Björnsson and Green [5]. They use a first quantized pure-spinor formalism [12] that exposes all conventional supersymmetry cancellations at the diagrammatic level. Enhanced cancellations between diagrams would be on top of these.

In principle, one would like to power count the supergravity integrands directly in order to understand the origin of cancellations. However, this is not feasible because the cancellations occur not only between different diagrams, but in particular between planar and nonplanar diagrams. We would need a global set of variables along with a formulation that exhibits integrand-level cancellations between diagrams. Unfortunately it is not known how to choose such variables for nonplanar diagrams. We therefore integrate the expressions in order to expose any potential ultraviolet cancellations.

There has been an attempt to explain the observed cancellations in  $\mathcal{N} = 4$  supergravity at three loops using standard supergravity symmetries by assuming the existence of a non-Lorentz covariant off-shell 16-supercharge harmonic superspace [13]. However, the required superspace predicts continued finiteness at three loops when matter is added to the theory, which is in contradiction to subsequent computations [14]. The finiteness therefore remains unexplained by standard-symmetry arguments.

What then might be behind enhanced cancellations? An analysis of the simpler but analogous case of half-maximal supergravity in  $D = 5$  at two loops points to the duality between color and kinematics as being responsible [15]. Unfortunately, this analysis is difficult to generalize to higher loops. It is therefore important to gather further data on the structure of ultraviolet divergences in supergravity theories, as we do in this paper.

Here we add to the available knowledge on the ultraviolet divergence structure of scattering amplitudes in supergravities by studying the three-loop half-maximal-supergravity critical dimension, which is defined as the lowest spacetime dimension where an ultraviolet divergence occurs. We give the divergence of half-maximal supergravity in the critical

dimension, as well as the divergence for an additional piece that allows us alter the field content to study theories with higher degrees of supersymmetry. To perform the computations, we analytically continue the loop momenta to a higher dimension  $D$  while keeping state-count parameters that allow us to tune the number of physical states. Different values for these parameters will give us access to different theories. Half-maximal and maximal supergravities, which in four dimensions are the  $\mathcal{N} = 4$  and  $\mathcal{N} = 8$  theories, respectively, have a natural definition in higher dimensions in terms of a dimensional continuation from a ten-dimensional theory. In fact, the maximal supergravity four-point integrand is unaltered through at least four loops as one changes the dimension [16].  $\mathcal{N} = 5$  supergravity, however, is a purely four-dimensional theory and is not well-defined in higher dimensions. Nevertheless, we can use four-dimensional values for the state-count parameters and search for additional cancellations with only the loop momenta analytically continued to higher dimensions.

As already mentioned, pure half-maximal  $\mathcal{N} = 4$  supergravity is three-loop finite in four dimensions [7]. The next dimension where there could be a divergence is  $D = 14/3$ , and we find that indeed half-maximal supergravity diverges here. We give the explicit value of the divergence and also collect various higher-dimensional one- and two-loop divergences. We give similar divergences for our additional piece that allows us access to theories with more supersymmetry.

This paper is organized as follows. In Sect. II, we summarize the methods that we use to obtain our results. In Sect. III, we give explicit values of higher-dimensional divergences at one loop. Then in Sect. IV, we give divergence results at two loops. Sect. V contains the main results of the paper: the divergences of half-maximal supergravity with and without matter at three loops in  $D = 14/3$ . We also discuss the three-loop divergences of theories with more supersymmetry through our additional piece. In Sect. VI, we give our concluding remarks.

## II. METHODS

In this section, we summarize the methods that we use to study the ultraviolet properties of supergravity theories. These methods have been previously discussed in detail in Ref. [14], so here we remain brief.

### A. Duality between color and kinematics

To obtain supergravity loop integrands, we use the duality between color and kinematics [8, 9]. This allows us to build supergravity integrands directly from simpler corresponding gauge-theory integrands.

The duality between color and kinematics is usually expressed in terms of diagrams with only three-point vertices. A generic color-dressed Yang-Mills amplitude at  $L$  loops may be written in terms of such diagrams as

$$\mathcal{A}_m^{(L)} = i^L g^{m-2+2L} \sum_{S_m} \sum_j \int \prod_{l=1}^L \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}. \quad (2.1)$$

The sum over  $j$  is over all trivalent diagrams; contributions containing four-point vertices are assigned to these diagrams by multiplying and dividing by appropriate propagators. There

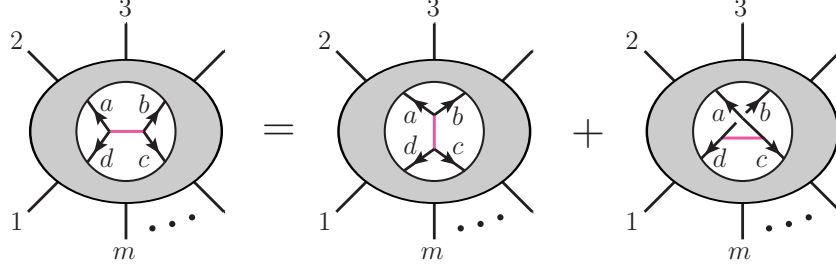


FIG. 1: The basic Jacobi relation between three diagrams for either color or numerator factors. The basic identity can be embedded into a diagram at any loop order.

is also a sum over the  $m!$  permutations of the external legs and a symmetry factor  $S_j$  to remove overcounts due to graph automorphisms. The numerator factor  $c_j$  is the color factor of graph  $j$  obtained by dressing each three-vertex with a structure constant  $\tilde{f}^{abc} = i\sqrt{2}f^{abc}$ . The numerator factor  $n_j$  is the kinematic numerator containing dependence on momenta, polarizations, spinors and, if a superspace formalism is used, Grassmann parameters. In the denominator,  $p_{\alpha_j}^2$  are the propagators of graph  $j$ .

The color-kinematics duality conjecture states that there exists a representation of amplitudes in the form of Eq. (2.1) such that the kinematic numerators  $n_j$  obey the same algebraic properties as color factors. In particular, the numerators obey the same Jacobi relations as the color factors of the same diagrams (see Fig. 1):

$$c_i = c_j - c_k \Rightarrow n_i = n_j - n_k. \quad (2.2)$$

The duality has been proven at tree level [17]; at loop level, it remains a conjecture, but we use it only where duality-satisfying representations have been explicitly constructed.

Once a duality-satisfying representation of a Yang-Mills amplitude is found, we obtain corresponding gravity amplitudes through the double-copy procedure by simply replacing the color factor in Eq. (2.1) with a duality-satisfying numerator of a second gauge theory:

$$c_j \rightarrow \tilde{n}_j. \quad (2.3)$$

After a trivial replacement of the coupling constant, this gives us the corresponding gravity amplitude,

$$\mathcal{M}_m^{(L)} = i^{L+1} \left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_{S_m} \sum_j \int \prod_{l=1}^L \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}. \quad (2.4)$$

The two numerators may belong to different Yang-Mills theories: By choosing different combinations of Yang-Mills theories, we obtain amplitudes in different (super)gravity theories. An important feature is that only one of the two copies  $n_j$  or  $\tilde{n}_j$  needs to satisfy the duality (2.2) for Eq. (2.4) to be valid [9, 18]. Furthermore, if it happens that one of the gauge-theory numerators vanishes for a certain diagram, then we do not need the corresponding numerator in the other gauge theory. In particular, for the one- and two-loop four-point amplitudes of  $\mathcal{N} \geq 4$  supergravity, the only required diagrams are those displayed in Fig. 2. All other diagrams have vanishing numerators in the maximally supersymmetric Yang-Mills component of the double-copy construction.

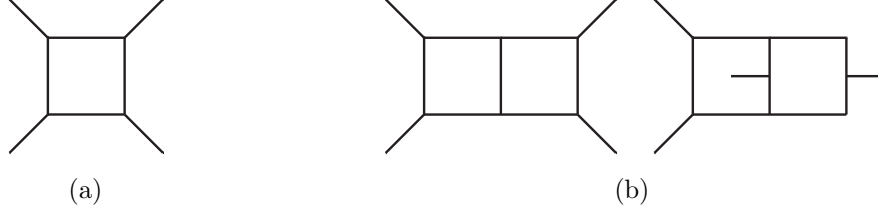


FIG. 2: The contributing diagrams at (a) one loop and (b) two loops in the double-copy construction of the four-point amplitude in supergravity theories with 16 or more supercharges.

## B. Construction of supergravity integrands

We now consider in more detail the specific constructions required for this paper. We start with half-maximal supergravity, which is a 16-supercharge theory. In four dimensions, it is the  $\mathcal{N} = 4$  supergravity theory [3]. We obtain half-maximal supergravity by taking one numerator in the double copy (2.4) to be from maximally supersymmetric Yang-Mills theory and the second from ordinary nonsupersymmetric Yang-Mills theory:

$$Q = 16 \text{ sugra} : (Q = 16 \text{ sYM}) \otimes (Q = 0 \text{ sYM}), \quad (2.5)$$

where  $Q$  counts the number of supercharges, and “sugra” and “sYM” are shorthands for, respectively, supergravity and super-Yang-Mills. By  $Q = 0$  sYM, we mean ordinary non-supersymmetric Yang-Mills theory. In  $D = 4$ ,  $Q = 16$  sYM is the  $\mathcal{N} = 4$  super-Yang-Mills theory. In this paper we calculate half-maximal supergravity divergences at one, two and three loops. In doing so, we leave as arbitrary a state-counting parameter  $D_s$  on the pure Yang-Mills side.  $D_s$  is the spin-state dimension of the internal gluons, i.e. the internal gluons have  $D_s - 2$  spin states. The loop momenta is still in  $D$  dimensions. Via dimensional reduction then, we are able to include  $(D_s - D)$  real scalars coupled to the nonsupersymmetric Yang-Mills theory in  $D$  dimensions. In half-maximal supergravity, this corresponds to including  $n_V = (D_s - D)$  matter vector multiplets in the theory.

We also calculate divergences of a piece that we call the “ $N_f$  piece”. It is obtained by taking a direct product of maximally supersymmetric Yang-Mills theory with nonsupersymmetric Yang-Mills theory coupled to non-chiral adjoint fermions, keeping only terms where each nonsupersymmetric Yang-Mills contribution contains at least one fermion loop. We define

$$N_f = n_f \text{Tr}[1], \quad (2.6)$$

where  $\text{Tr}[1]$  gives the dimensionality of the gamma matrices, and  $n_f$  is the number of fermion flavors. The parameter  $N_f$  is thus the number of fermionic states of one of the gauge theories from which the gravity theory is composed, not counting color degrees of freedom.  $Q = 16$  super-Yang-Mills theory contains 16 states in any dimension, so the total number of states in the corresponding supergravity theory is  $16(N_f + D_s - 2)$ . We again allow for an arbitrary number of scalars in the  $N_f$  piece through the parameter  $D_s$ . The  $N_f$  piece is the difference between two supergravity calculations:

$$\begin{aligned} N_f \text{ piece} : & (Q = 16 \text{ sYM}) \otimes (Q = 0 \text{ sYM} + N_f \text{ fermion states}) \\ & \ominus (Q = 16 \text{ sYM}) \otimes (Q = 0 \text{ sYM}), \end{aligned} \quad (2.7)$$

where  $(Q = 0 \text{ sYM} + N_f \text{ fermion states})$  represents pure nonsupersymmetric Yang-Mills theory coupled to non-chiral adjoint fermions. While this formula is valid for any states in the

theory, in this paper we will consider only external states constructed from any state on the  $Q = 16$  super-Yang-Mills side and only gluons from the other side. There is no restriction on the states circulating in the loops. This decomposition allows us to extend half-maximal supergravity results to theories with higher degrees of supersymmetry by controlling the number of scalars and non-chiral fermions on the non-maximal super-Yang-Mills side.

For certain choices of the parameters  $D_s$  and  $N_f$  and loop integration dimension  $D$ , the obtained theories correspond to well-known supergravity theories. For example, in Ref. [6], we promoted  $\mathcal{N} = 4$  supergravity in four dimensions to  $\mathcal{N} = 5$  supergravity in  $D = 4$  by including the  $N_f$  piece and choosing  $D_s = 4$  and  $N_f = 2$ . The end result is equivalent to taking a direct product of  $\mathcal{N} = 4$  super-Yang-Mills theory and  $\mathcal{N} = 1$  super-Yang-Mills theory. Including both the half-maximal piece and the  $N_f$  piece for divergences in this paper will allow us to easily examine cases besides half-maximal supergravity, most notably maximally supersymmetric supergravity. Many results in the maximal theory are already known, so it provides useful checks on our results. For the three-loop case, we also look at  $\mathcal{N} = 5$  supergravity by keeping the state counts at their four-dimensional values. For generic values of the state-counting parameters, this construction amounts to analytic continuations of the well-known supergravity theories. We note that our construction is not valid for four-dimensional  $\mathcal{N} = 6$  supergravity. We can obtain this from a dimensional reduction of  $Q = 24$  or  $\mathcal{N} = 3$  supergravity in  $D = 6$ , but that would require six-dimensional chiral fermions for the  $N_f$  contributions, which we do not incorporate in the present calculation.

In our construction we use the color-kinematics duality-satisfying representations on the maximal  $Q = 16$  super-Yang-Mills side [9]. Since only one copy in Eq. (2.4) needs to satisfy the duality (2.2), we are free to use any other convenient representation on the nonsupersymmetric side of the double copy. At four points, the one- and two-loop amplitudes are especially easy to obtain because the maximal super-Yang-Mills kinematic numerators are independent of loop momenta [15, 19]. This means that the corresponding supergravity amplitudes are linear combinations of gauge-theory ones even after integration. Three loops is more complicated. At three loops, we use Feynman rules for the non-maximally supersymmetric side. While this may seem inefficient, it is a good choice for our purposes. We only need to keep those Feynman diagrams that contain color factors corresponding to nonvanishing numerators on the maximally supersymmetric Yang-Mills side. We also use Feynman diagrams as a cross check on our one- and two-loop results below.

### C. Extraction of ultraviolet divergences

After using the double-copy formula (2.4) to construct the integrands of the half-maximal and  $N_f$  pieces, we must integrate to extract the ultraviolet divergence. Our procedure is described in detail in Ref. [14]; here we briefly summarize it.

Since we are only interested in the ultraviolet behavior, we series expand power-divergent integrals in large internal loop momenta, or equivalently small external momenta [20]. In this expansion, only the logarithmically divergent integrals contribute to the ultraviolet divergence, so we drop all other pieces. After removing subdivergences, the divergences of logarithmically divergent integrals are proportional to numbers with no dependence on external momenta. We may therefore set the external momenta in the resulting integrals to zero, converting the integrals to vacuum integrals that are much easier to evaluate.

The resulting vacuum integrals are badly infrared divergent, so we need regulators that separate these singularities from the ultraviolet ones that we are interested in obtaining.

For the ultraviolet divergences we use dimensional reduction [21]. A particularly good choice of infrared regulator is to assign all propagators a uniform mass at the start of the calculation [22]. In many cases, this uniformity allows for cancellations at lower loop orders to feed into the calculation, eliminating the need to subtract subdivergences integral by integral [20]. For the three-loop cases in  $D = 14/3$ , none of the individual integrals actually have subdivergences, so subtractions are not required. However, for the two-loop cases in six dimensions examined here, subdivergence subtractions are required even with a uniform mass due to quadratic subdivergences, which apparently break the feed-through of lower-loop cancellations. An advantage of explicitly subtracting subdivergences is that it allows us to avoid potential subtleties associated with evanescent operators that might occur when using counterterms to remove lower-loop subdivergences [23].

Once we have the logarithmically-divergent, mass-regulated vacuum integrals, we reduce to a basis of master integrals using **FIRE** [10], which implements integration-by-parts identities using the Laporta algorithm [24]. In  $D = 4$  the master integrals are available in the literature since they correspond to ones used in the calculations of the three- and four-loop QCD  $\beta$  functions [25]. We also perform the integrals explicitly using Mellin-Barnes techniques and resummation of residues [26]. The three-loop integrals in  $D = 14/3$  required for our computation are evaluated this way.

### III. ONE-LOOP DIVERGENCES

At one loop, all pure supergravities are finite in  $D = 4$  [27]. They are also finite in  $D = 6$  because the required  $R^3$  counterterm violates supersymmetry identities of a four-dimensional subspace [28]. Consistent with this, we find that both the half-maximal piece and the  $N_f$  piece are separately finite in these dimensions. With matter, however, half-maximal supergravity is divergent in both  $D = 4$  and  $D = 6$ . Refs. [14, 29] contain the explicit forms of the divergences for external matter in four dimensions; Ref. [14] contains divergences for external matter in six dimensions.

In this section we will focus on  $D = 8$ , which is the one-loop critical dimension where divergences first occur when there are no matter multiplets. We first review  $D = 8$  one-loop results in half-maximal supergravity and give new results for the  $N_f$  piece. The divergence for half-maximal supergravity in  $D = 8$  was first given in Ref. [30]. Divergences involving external matter states in eight dimensions can be found in Ref. [14].

#### A. Eight dimensions

In eight dimensions, the divergence for pure half-maximal supergravity with internal matter is [15, 30]

$$\begin{aligned} \mathcal{M}_{Q=16}^{(1)}(1, 2, 3, 4) \Big|_{D=8 \text{ div.}} &= -\frac{1}{\epsilon} \frac{1}{(4\pi)^4} \left(\frac{\kappa}{2}\right)^4 st A_{Q=16}^{\text{tree}}(1, 2, 3, 4) \\ &\times \left[ \frac{238 + D_s}{360} (F_1 F_2 F_3 F_4) + \frac{D_s - 50}{288} (F_1 F_2) (F_3 F_4) \right] \\ &+ \text{cyclic}(2, 3, 4), \end{aligned} \quad (3.1)$$

where  $\epsilon = (8 - D)/2$  is the usual dimensional regularization parameter,  $A_{Q=16}^{\text{tree}}(1, 2, 3, 4)$  is the maximal super-Yang-Mills four-point tree amplitude, and  $\text{cyclic}(2, 3, 4)$  represents the

cyclic permutations over legs 2, 3 and 4. We use the Mandelstam invariants,

$$s = (k_1 + k_2)^2, \quad t = (k_2 + k_3)^2, \quad u = (k_1 + k_3)^2. \quad (3.2)$$

Here

$$(F_i F_j) \equiv F_i^{\mu\nu} F_{j\mu\nu}, \quad (F_i F_j F_k F_l) \equiv F_i^{\mu\nu} F_{j\nu\rho} F_k^{\rho\sigma} F_{l\sigma\mu}, \quad (3.3)$$

where the linearized polarization field strength of each leg  $j$  is

$$F_j^{\mu\nu} \equiv i(k_j^\mu \varepsilon_j^\nu - k_j^\nu \varepsilon_j^\mu). \quad (3.4)$$

The divergence can also be generated by the operator [15, 30],

$$\begin{aligned} \mathcal{O} = & -\frac{1}{\epsilon} \frac{1}{(4\pi)^4} \frac{1}{11520} [(-126 + 3D_s)T_1 + (1968 - 24D_s)T_2 + (-252 + 6D_s)T_3 \\ & + (8 - 4D_s)T_4 + 3840T_5 - 1920T_6 + (-3776 - 32D_s)T_7], \end{aligned} \quad (3.5)$$

where

$$\begin{aligned} T_1 &= (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma})^2, \\ T_2 &= R_{\mu\nu\rho\sigma} R^{\mu\nu\rho}{}_\lambda R_{\gamma\delta\kappa}{}^\sigma R^{\gamma\delta\kappa\lambda}, \\ T_3 &= R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\lambda\gamma} R^{\lambda\gamma}{}_{\delta\kappa} R^{\rho\sigma\delta\kappa}, \\ T_4 &= R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\lambda\gamma} R^{\rho\lambda}{}_{\delta\kappa} R^{\sigma\gamma\delta\kappa}, \\ T_5 &= R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\lambda\gamma} R^{\rho}{}_\delta{}^\lambda{}_\kappa R^{\sigma\delta\gamma\kappa}, \\ T_6 &= R_{\mu\nu\rho\sigma} R^\mu{}_\lambda{}^\rho{}_\gamma R^\lambda{}_\delta{}^\gamma{}_\kappa R^{\nu\delta\sigma\kappa}, \\ T_7 &= R_{\mu\nu\rho\sigma} R^\mu{}_\lambda{}^\rho{}_\gamma R^\lambda{}_\delta{}^\nu{}_\kappa R^{\gamma\delta\sigma\kappa}. \end{aligned} \quad (3.6)$$

We note that on shell the combination

$$-\frac{T_1}{16} + T_2 - \frac{T_3}{8} - T_4 + 2T_5 - T_6 + 2T_7, \quad (3.7)$$

is a total derivative [30], so there is freedom in rewriting Eq. (3.5). For four-dimensional external states in an MHV-helicity configuration, we can write the divergence in terms of four-dimensional spinor helicity [15]:

$$\mathcal{M}_{Q=16}^{(1)}(1^-, 2^-, 3^+, 4^+) \Big|_{D=8 \text{ div.}} = \frac{i}{\epsilon} \frac{1}{(4\pi)^4} \left(\frac{\kappa}{2}\right)^4 \frac{58 + D_s}{180} \langle 12 \rangle^4 [34]^4. \quad (3.8)$$

For the  $N_f$  piece, we have

$$\begin{aligned} \mathcal{M}_{N_f}^{(1)}(1, 2, 3, 4) \Big|_{D=8 \text{ div.}} = & -\frac{1}{\epsilon} \frac{1}{(4\pi)^4} N_f \left(\frac{\kappa}{2}\right)^4 st A_{Q=16}^{\text{tree}}(1, 2, 3, 4) \\ & \times \left[ \frac{7}{180} (F_1 F_2 F_3 F_4) - \frac{1}{72} (F_1 F_2) (F_3 F_4) \right] + \text{cyclic}(2, 3, 4). \end{aligned} \quad (3.9)$$

This can also be generated by the operator,

$$\mathcal{O} = \frac{1}{\epsilon} \frac{1}{(4\pi)^4} N_f \frac{1}{23040} [21T_1 - 288T_2 + 42T_3 - 8T_4 - 480T_5 + 240T_6 + 416T_7]. \quad (3.10)$$



For four-dimensional external states in an MHV-helicity configuration, we have

$$\mathcal{M}_{N_f}^{(1)}(1^-, 2^-, 3^+, 4^+) \Big|_{D=8 \text{ div.}} = \frac{i}{\epsilon} \frac{1}{(4\pi)^4} \left(\frac{\kappa}{2}\right)^4 N_f \frac{11}{720} \langle 12 \rangle^4 [34]^4. \quad (3.11)$$

As a check on these results, we obtain the result for maximal supergravity by setting  $D_s = 10$  and  $N_f = 8$ . This gives

$$\begin{aligned} \mathcal{M}_{Q=32}^{(1)}(1, 2, 3, 4) \Big|_{D=8 \text{ div.}} &= -\frac{1}{\epsilon} \frac{1}{(4\pi)^4} \left(\frac{\kappa}{2}\right)^4 stA_{Q=16}^{\text{tree}}(1, 2, 3, 4) \\ &\quad \times \left[ (F_1 F_2 F_3 F_4) - \frac{1}{4} (F_1 F_2)(F_3 F_4) \right] + \text{cyclic}(2, 3, 4) \\ &= -\frac{i}{\epsilon} \frac{1}{(4\pi)^4} \left(\frac{\kappa}{2}\right)^4 \frac{1}{2} \left[ stA_{Q=16}^{\text{tree}}(1, 2, 3, 4) \right]^2, \end{aligned} \quad (3.12)$$

in agreement with the known result [31]. The gravity operator corresponding to this divergence is [30]

$$\mathcal{O} = \frac{1}{\epsilon} \frac{1}{(4\pi)^4} \frac{1}{64} [T_1 - 16T_2 + 2T_3 - 32T_5 + 16T_6 + 32T_7]. \quad (3.13)$$

For the MHV four-graviton configuration in terms of four-dimensional spinor helicity, this is

$$\mathcal{M}_{Q=32}^{(1)}(1^-, 2^-, 3^+, 4^+) \Big|_{D=8 \text{ div.}} = \frac{i}{\epsilon} \frac{1}{(4\pi)^4} \left(\frac{\kappa}{2}\right)^4 \frac{1}{2} \langle 12 \rangle^4 [34]^4. \quad (3.14)$$

#### IV. TWO-LOOP DIVERGENCES

It has been known since the early days of supergravity that all pure theories without matter are two-loop finite in  $D = 4$  because no valid counterterms exist [28, 32]. In five dimensions, however, an apparently valid  $R^4$  counterterm exists for half-maximal supergravity under supersymmetry and duality-symmetry constraints [13, 14, 33]. Nevertheless, a string-theory calculation [11] and an explicit field-theory calculation [15] demonstrate that half-maximal supergravity is in fact ultraviolet finite at two loops in  $D = 5$ . The cancellations are not visible diagram by diagram in any covariant local diagrammatic formalism, so this case is an example of enhanced cancellations [6]. Interestingly, the cancellations can be instead understood by relating the supergravity amplitudes to Yang-Mills amplitudes involving color tensors that are forbidden to appear in divergences [15]. A key point in this analysis is that it captures cancellations between diagrams, particularly between the planar and nonplanar sectors.

The two-loop critical dimension for pure half-maximal supergravity is  $D_c = 6$ . The explicit divergence including internal matter vector multiplets was given in Ref. [15]. In this section we review this result and give the new result of the divergence for the  $N_f$  piece. With external matter states, half-maximal supergravity diverges in  $D = 4, 5$  and 6; explicit forms can be found in Ref. [14].

### A. Six dimensions

In  $D = 6$ , the divergence in half-maximal supergravity with internal-matter multiplets is [15]

$$\begin{aligned} \mathcal{M}_{Q=16}^{(2)}(1, 2, 3, 4) \Big|_{D=6 \text{ div.}} &= \frac{1}{(4\pi)^6} \left(\frac{\kappa}{2}\right)^6 st A_{Q=16}^{\text{tree}}(1, 2, 3, 4) \\ &\times \left\{ \left( \frac{(D_s - 6)(26 - D_s)}{576\epsilon^2} + \frac{19D_s - 734}{864\epsilon} \right) \right. \\ &\quad \times \left[ s(F_1 F_2)(F_3 F_4) + t(F_1 F_4)(F_2 F_3) + u(F_1 F_3)(F_2 F_4) \right] \\ &\quad \left. + \frac{48D_s - 1248}{864\epsilon} \left[ u(F_1 F_2 F_3 F_4) + t(F_1 F_3 F_4 F_2) + s(F_1 F_4 F_2 F_3) \right] \right\}. \end{aligned} \quad (4.1)$$

The number of vector matter multiplets is given by  $n_V = D_s - 6$ . The result for pure half-maximal supergravity is obtained by setting  $D_s = 6$ , and we find that even with no matter multiplets, the theory still diverges. The  $1/\epsilon^2$  divergence vanishes in this case because there are no one-loop subdivergences in the pure theory.

We can simplify the expressions a bit by restricting the external states to four-dimensional helicity states. For the MHV configuration, this gives the divergence [15],

$$\begin{aligned} \mathcal{M}_{Q=16}^{(2)}(1^-, 2^-, 3^+, 4^+) \Big|_{D=6 \text{ div.}} &= -\frac{i}{(4\pi)^6} \left(\frac{\kappa}{2}\right)^6 s \langle 1 2 \rangle^4 [3 4]^4 \\ &\times \left( \frac{(D_s - 6)(26 - D_s)}{576\epsilon^2} + \frac{19D_s - 734}{864\epsilon} \right). \end{aligned} \quad (4.2)$$

The  $N_f$  piece also diverges in six dimensions. Its form is

$$\begin{aligned} \mathcal{M}_{N_f}^{(2)}(1, 2, 3, 4) \Big|_{D=6 \text{ div.}} &= \frac{1}{(4\pi)^6} \left(\frac{\kappa}{2}\right)^6 N_f st A_{Q=16}^{\text{tree}}(1, 2, 3, 4) \left\{ \left( \frac{6 - D_s}{288\epsilon^2} + \frac{21D_s + 62}{3456\epsilon} \right) \right. \\ &\quad \times \left[ s(F_1 F_2)(F_3 F_4) + t(F_1 F_4)(F_2 F_3) + u(F_1 F_3)(F_2 F_4) \right] \\ &\quad \left. + \frac{3D_s - 14}{144\epsilon} \left[ u(F_1 F_2 F_3 F_4) + t(F_1 F_3 F_4 F_2) + s(F_1 F_4 F_2 F_3) \right] \right\}. \end{aligned} \quad (4.3)$$

For the MHV external helicity configuration, we find

$$\mathcal{M}_{N_f}^{(2)}(1^-, 2^-, 3^+, 4^+) \Big|_{D=6 \text{ div.}} = \frac{i}{(4\pi)^6} \left(\frac{\kappa}{2}\right)^6 N_f s \langle 1 2 \rangle^4 [3 4]^4 \left( \frac{D_s - 6}{288\epsilon^2} - \frac{21D_s + 62}{3456\epsilon} \right). \quad (4.4)$$

As noted earlier, different choices of  $N_f$  and  $D_s$  do not necessarily correspond to supersymmetric theories. However, the choice  $D_s = 10$  and  $N_f = 8$  does correspond to maximal supergravity obtained by dimensional reduction of  $\mathcal{N} = 1$ ,  $D = 10$  supergravity to  $D = 6$ . With this choice, we find that by adding together the contributions in Eqs. (4.1) and (4.3), the total divergence vanishes. This matches a previous calculation that shows that at two loops, maximal supergravity is ultraviolet finite for  $D < 7$  [34].

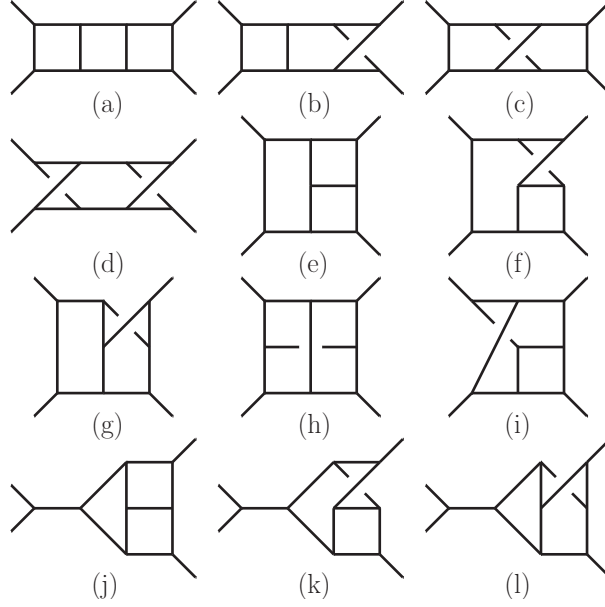


FIG. 3: The contributing diagrams to the three-loop four-point amplitudes in theories with 16 or more supercharges.

## V. THREE-LOOP DIVERGENCES

In Ref. [7], the three-loop four-point amplitudes of pure half-maximal supergravity in  $D = 4$  (usually called  $\mathcal{N} = 4$  supergravity) were shown to be ultraviolet finite. The contributing diagrams are shown in Fig. 3. The finiteness holds despite the existence of an apparently valid counterterm under supersymmetry and duality-symmetry constraints [13, 14, 33]. The maximal-cut analysis of Ref. [6] shows that individual diagrams necessarily diverge, so the finiteness is a prime example of enhanced cancellations [6].

An interesting question is to identify the critical dimension of the three-loop amplitudes corresponding to the lowest dimension with ultraviolet divergences. The next possible dimension above  $D = 4$  where divergences can occur for this amplitude is  $D = 14/3$ . As we now show, this is indeed the critical dimension for pure half-maximal supergravity at three loops. We provide the explicit form of the divergence, as well as the form of the divergence of the  $N_f$  piece. We also provide results in  $D = 14/3$  for half-maximal supergravity coupled to matter. Results including matter in  $D = 4$  can be found in Ref. [14].

### A. $D = 14/3$ dimensions with no matter

Following the steps described in Sect. II, we find that the divergence for pure half-maximal supergravity in  $D = 14/3$  is

$$\mathcal{M}_{Q=16}^{(3)}(1, 2, 3, 4) \Big|_{D=14/3 \text{ div.}} = -\frac{1}{(4\pi)^7} \left(\frac{\kappa}{2}\right)^8 \frac{9(10056 - 2546D_s + 99D_s^2)}{61600\epsilon} \times \Gamma^3\left(\frac{1}{3}\right) st A_{Q=16}^{\text{tree}}(1, 2, 3, 4) \mathcal{P}, \quad (5.1)$$

where

$$\mathcal{P} = \sum_{S_4} \left( \frac{1}{64} stu \varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_4 + \frac{1}{8} t \varepsilon_1 \cdot \varepsilon_2 k_1 \cdot \varepsilon_3 (-3u k_1 \cdot \varepsilon_4 + (s+3t) k_2 \cdot \varepsilon_4) \right. \\ \left. + \frac{1}{12} t k_2 \cdot \varepsilon_1 k_3 \cdot \varepsilon_2 (-4k_1 \cdot \varepsilon_3 k_1 \cdot \varepsilon_4 - 3k_2 \cdot \varepsilon_3 k_1 \cdot \varepsilon_4 + k_2 \cdot \varepsilon_3 k_2 \cdot \varepsilon_4) \right), \quad (5.2)$$

and the sum is over all 24 permutations of the external legs. For the  $N_f$  piece, we find

$$\mathcal{M}_{N_f}^{(3)}(1, 2, 3, 4) \Big|_{D=14/3 \text{ div.}} = - \frac{1}{(4\pi)^7} \left( \frac{\kappa}{2} \right)^8 \frac{3(32136 - 7738D_s + 535D_s^2)}{246400\epsilon} \\ \times N_f \Gamma^3\left(\frac{1}{3}\right) st A_{Q=16}^{\text{tree}}(1, 2, 3, 4) \mathcal{P}. \quad (5.3)$$

It is interesting that the potential term proportional to  $N_f^2$  vanishes.

By setting  $D_s = 10$  and  $N_f = 8$ , we obtain maximally supersymmetric supergravity continued to  $D = 14/3$  dimensions. With this choice of parameters, by adding together Eqs. (5.1) and (5.3), we find that the divergence cancels, in agreement with the known finiteness of the maximally supersymmetric theory at three loops for  $D < 6$  [35]. It is also interesting to look at the analytic continuation of  $\mathcal{N} = 5$  theory using four-dimensional state counts. Setting  $D_s = 4$  and  $N_f = 2$ , we find that the amplitude is still divergent, so there are no additional cancellations beyond the pure half-maximal theory. It is interesting to note that the sum of the two divergences in Eqs. (5.1) and (5.3) vanishes when we take  $D_s = 6$  and  $N_f = 4$ . Presumably this cancellation is not accidental. These contributions give part of the result for four-dimensional  $\mathcal{N} = 6$  supergravity analytically continued to  $D = 14/3$ ; the pieces arising from dimensional reduction of the chirality projector on fermions on the  $N_f$  side of the double copy is missing from our construction. It would be interesting to obtain this piece as well to see if  $\mathcal{N} = 6$  supergravity with four-dimensional states is three-loop finite when the loop momenta are analytically continued to  $D = 14/3$ .

It is also interesting to look at the divergences when the external states are restricted to live in a four-dimensional subspace where standard helicity states can be used. Focusing on the gauge-invariant factor  $\mathcal{P}$ , we write helicity terms for the nonsupersymmetric side of the double copy using four-dimensional spinor helicity:

$$\mathcal{P}_{--++} = 0, \\ \mathcal{P}_{-+++} = -\frac{1}{8} s^2 t^2 \frac{[2\,4]^2}{[1\,2]\langle 2\,3\rangle\langle 3\,4\rangle[4\,1]}, \\ \mathcal{P}_{++++} = \frac{5}{4} stu \frac{[1\,2][3\,4]}{\langle 1\,2\rangle\langle 3\,4\rangle}. \quad (5.4)$$

where the subscripts on  $\mathcal{P}$  indicate the helicity choices in an all-outgoing convention. The vanishing of the MHV configuration is consistent with the fact that the pure Yang-Mills counterterm is an  $F^3$  operator at three loops in  $D = 14/3$ :  $F^3$  cannot generate the MHV configuration at four points.

## B. $D = 14/3$ dimensions with matter

Half-maximal supergravity at three loops in  $D = 14/3$  coupled to matter also diverges as one might expect. The divergences of this theory in  $D = 4$  were already discussed in

Ref. [14]. For the amplitude with two external graviton multiplets and two external vector multiplets, we find

$$\begin{aligned}
\mathcal{M}_{Q=16}^{(3)}(1_H, 2_H, 3_V, 4_V) \Big|_{D=14/3 \text{ div.}} &= \frac{1}{(4\pi)^7} \left(\frac{\kappa}{2}\right)^8 \frac{1}{985600\epsilon} \Gamma^3\left(\frac{1}{3}\right) s t A_{Q=16}^{\text{tree}}(1, 2, 3, 4) \\
&\times \left( 5(693D_s^2 - 67922D_s + 443136) s^2 (2k_1 \cdot \varepsilon_2 k_2 \cdot \varepsilon_1 - s \varepsilon_1 \cdot \varepsilon_2) \right. \\
&\quad - 48(693D_s^2 + 1678D_s - 74688) t u k_1 \cdot \varepsilon_2 k_2 \cdot \varepsilon_1 \\
&\quad + 12(1881D_s^2 + 3626D_s - 195816) s t u \varepsilon_1 \cdot \varepsilon_2 \\
&\quad \left. + 1080(11D_s^2 + 6D_s - 1032) s \right. \\
&\quad \left. \times (s k_3 \cdot \varepsilon_1 k_3 \cdot \varepsilon_2 - t k_1 \cdot \varepsilon_2 k_3 \cdot \varepsilon_1 - u k_2 \cdot \varepsilon_1 k_3 \cdot \varepsilon_2) \right),
\end{aligned} \tag{5.5}$$

where the subscript “H” indicates a particle belonging to a graviton multiplet and “V” indicates a particle belonging to a matter vector multiplet. For four external matter particles belonging to the same multiplet, we find the divergence,

$$\begin{aligned}
\mathcal{M}_{Q=16}^{(3)}(1_V, 2_V, 3_V, 4_V) \Big|_{D=14/3 \text{ div.}} &= -\frac{1}{(4\pi)^7} \left(\frac{\kappa}{2}\right)^8 \Gamma^3\left(\frac{1}{3}\right) s^2 t^2 u A_{Q=16}^{\text{tree}}(1, 2, 3, 4) \\
&\times \frac{3}{123200\epsilon} (8811D_s^2 + 104656D_s - 1569516).
\end{aligned} \tag{5.6}$$

Finally for external matter from different multiplets, we have

$$\begin{aligned}
\mathcal{M}_{Q=16}^{(3)}(1_{V_1}, 2_{V_1}, 3_{V_2}, 4_{V_2}) \Big|_{D=14/3 \text{ div.}} &= \frac{1}{(4\pi)^7} \left(\frac{\kappa}{2}\right)^8 \frac{1}{123200\epsilon} \Gamma^3\left(\frac{1}{3}\right) s^2 t A_{Q=16}^{\text{tree}}(1, 2, 3, 4) \\
&\times \left( 10(495D_s^2 - 28630D_s + 168576) s^2 - 3(4587D_s^2 - 60548D_s + 38748) t u \right),
\end{aligned} \tag{5.7}$$

where the “V<sub>1</sub>” and “V<sub>2</sub>” subscripts indicate that the particles belong to different matter multiplets.

## VI. CONCLUSIONS

Recently a new type of ultraviolet cancellation in multiloop supergravity amplitudes called enhanced cancellation [6] was uncovered. The defining characteristic of these cancellations is that they cannot be made manifest in a covariant local diagrammatic formalism. This makes the cancellations nontrivial to study. At present there is no explanation for these cancellations based on supersymmetry and the standard symmetries of supergravity [13, 14, 33]. There is some evidence at one and two loops that the duality between color and kinematics [8, 9] might be responsible for the cancellations [15]. However, it is nontrivial to extend this analysis to higher loops. It therefore is important to collect as much data as possible on the ultraviolet structure of supergravity theories.

In this paper we provided new information on the divergence structure of supergravity theories by analytically continuing the amplitudes to higher dimensions. In particular, we

analytically continued half-maximal supergravity to  $D = 14/3$ , corresponding to the lowest dimension where the three-loop four-point amplitude could diverge. Indeed, we found a divergence and presented its explicit form. We also provided the divergences in this number of dimensions with the addition of matter multiplets. Another interesting case that we studied in  $D = 14/3$  is the analytic continuation of the three-loop four-point amplitude of  $\mathcal{N} = 5$  supergravity to this dimension, keeping the parameters that control the number of states as free parameters. Again we found no sensible solution that makes the amplitude finite. In any case, the existence of enhanced cancellations in supergravity theories shows that we have much more to learn about their ultraviolet structure.

## Acknowledgments

We thank John Joseph Carrasco, Henrik Johansson and Radu Roiban for discussions. This work was supported by the Department of Energy under Award Number DE-SC0009937. We also thank the Danish Council for Independent Research. We gratefully acknowledge Mani Bhaumik for his generous support. We also thank Academic Technology Services at UCLA for computer support.

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